

A Thermodynamic Sector of Quantum Gravity

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Abstract

The connection between gravity and thermodynamics is explored. Examining a perfect fluid in gravitational equilibrium we find that the entropy is extremal only if Einstein's equations are satisfied. Conversely, one can derive part of Einstein's equations from ordinary thermodynamical considerations. This allows the theory of this system to be recast in such a way that a sector of general relativity is purely thermodynamical and should not be quantized.

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I. INTRODUCTION

Einstein's equations relate the geometry of space-time (as expressed by the Einstein tensor $G_{\mu\nu}$) to the distribution of matter (as expressed by the stress-energy tensor $T_{\mu\nu}$). Many connections between these equations and thermodynamics have been emerging in recent years. Einstein's equations imply that black holes possess a temperature[1], and their horizon behaves like an entropy [2]. In dynamical gravitating systems, critical phenomena has been observed [3].

In these examples, the horizon seems to play a central role. In this paper, we will explore the connection between thermodynamics and gravity by examining a gravitating system which does not possess a horizon. In particular, a spherically symmetric perfect fluid in equilibrium will be considered. We find that the connection between the thermodynamics of this system and general relativity is extremely deep - in fact, it is an identity. By extremizing the total entropy of the matter while keeping the total particle number and energy fixed, one finds that the most probable configuration is one which obeys a linear combination of the two independent Einstein's equations (the other 8 components of Einstein's equations are trivially satisfied). The fact that the entropy of the system is only an extremum when Einstein's equations are satisfied can be considered as a counterpart to black hole thermodynamics for systems without horizons.

However, this close connection between thermodynamics and Einstein's equations appear somewhat puzzling - there does not appear to be any reason *a priori* why the entropy should only be an extremum for Einstein's equations. This leads us to suggest that this can be viewed as evidence that Einstein's equations represent an effective theory. In particular, for this system, one can derive one of Einstein's equations from thermodynamics.

As a result one need not assume all of Einstein's equations for this system. Rather, one only needs to assume a single equation, and the remaining equations will be automatically satisfied due to thermodynamical considerations. One can therefore view one of the two independent Einstein's equations as having no physical content - it is just a restatement of the laws of equilibrium statistical mechanics in curved space.

If one adopts such a position, then one need not quantize the equation which is derived from thermodynamics. In fact, it would make as much sense to quantize this equation as it would be to quantize the laws of hydrodynamics or the laws of thermodynamics themselves.

One does not quantize the Navier-Stokes equation or the first law of thermodynamics; rather, one quantizes the underlying equations governing the individual molecules which make up the fluid or thermodynamic system.

The notion that general relativity is an effective theory which should not be quantized directly was strongly supported by the derivation given by Jacobson [9]. There, it was shown that if there exists a microscopic theory where causal horizons have an entropy proportional to their area, then Einstein's equations will automatically hold. In the present work, no assumption need be made about the underlying microscopic theory of geometry. In fact, here, the quantum theory is simply the ordinary quantum theory of the matter.

The thermodynamics and statistical mechanics of gravitating systems have been studied using toy models, mean field theories, and numerical methods[4]. In some of these toy models, the entropy is not a global extremum[5] while in other models, and using other methods it is[6]. For example, if one couples a gravitational system to an infinite and external thermal reservoir, then the negative specific heat of the gravitational system will make thermal equilibrium impossible. Here, there is no external heat bath or containment box, and furthermore, our study is fully relativistic. We find that the total entropy is an extremum when Einstein's equation is satisfied. In general, the study of thermodynamics for systems with long range interactions can be problematic. In fact, an entirely new definition of entropy has been used to attempt to deal with these systems[7] because it was not believed that one can properly analyze these systems using conventional thermodynamics. Here, we see that using the equivalence principle to define all thermodynamic quantities locally allows us to completely analyze the system without any need to make approximations or go outside traditional thermodynamics. A more exhaustive analysis of these issues will be given elsewhere [8].

In the next section, we will introduce the system under consideration and show that local validity of the first law of thermodynamics is equivalent to the vanishing of the divergence of the stress-energy tensor. This is not at all surprising but the method of derivation is instructive. We will then derive the conditions for maximal entropy and show that it is equivalent to one of Einstein's equations. In Section III we show that for this system one can recast the equations of general relativity into a different form where one equation is purely thermodynamical and the other equation gives the physical theory. We conclude with a discussion on the possible implications of this result for the quantum theory of gravity.

Note: I have recently become aware of a related work by Harrison et. al. [15] in their study of gravitating fluids. There, the entropy of the system is assumed to be zero, however the methodology is somewhat similar. They assume the correct form for the proper volume in the space time of a perfect fluid. They then derive the Oppenheimer-Volkoff equation by minimizing the Arnowitt-Desser-Misner mass. There too, the authors appear puzzled by this result, although they speculate that that it may be due to the simplified geometry.

II. ENTROPY AND EINSTEIN'S EQUATIONS

Initially, we will not assume any of Einstein's equations. However, we assume a metric theory of gravity, which will allow us to exploit the equivalence principle. Since we are considering a spherically symmetric system, the metric takes the familiar form

$$ds^2 = -e^{2\Phi} dt^2 + e^{2\Lambda} dr^2 + r^2 d\Omega^2 \quad . \quad (1)$$

Since we are considering a system in equilibrium the metric is independent of time. We will not specify the metric functions – in particular, we will not demand that this metric satisfy Einstein's equations. For matter, we assume the stress-energy tensor of the perfect fluid, which is given in terms of the energy density $\rho(r)$ and radial pressure $p(r)$ by

$$T^{\mu\nu} = (\rho + p)u^\mu(r)u^\nu(r) + p(r)g^{\mu\nu} \quad (2)$$

where $u^\mu(r)$ is the 4-velocity of the fluid and $g^{\mu\nu}$ is the metric. It is worth noting that spherical symmetry would follow automatically for a perfect fluid in equilibrium (i.e. stationary) if we were to assume asymptotic flatness.

We make no assumptions about specific equations of states for the matter, but we will assume that in the absence of gravity, the fluid has no unscreened long-range interactions and is therefore *extensive*. In other words, the entropy, particle number and energy scale as the size of the system when the intensive variables (temperature, pressure and chemical potential) are held fixed. In virtually all applications of thermodynamics this is assumed to be the case (at least implicitly), for without it, taking the thermodynamic limit becomes problematic and also, no canonical ensemble would exist[4]. This assumption of extensivity is only assumed to hold true in the absence of gravity; once we include gravity, it will no longer hold globally. In the absence of gravity, extensivity plus the first law imply the

Gibbs-Duhem relation [10]

$$\rho = Ts - p + \mu n \quad (3)$$

where s, ρ, n are the entropy density, energy density and particle number density of the matter as measured in its proper reference frame, and T, μ and p are the local temperature, chemical potential and pressure.

A gravitating perfect fluid obeys the three equations[11]

$$\frac{dp}{dr} = -(\rho + p) \frac{d\Phi}{dr} \quad (4)$$

$$e^{-2\Lambda} = 1 - 2m(r)/r \quad (5)$$

$$\frac{d\Phi}{dr} = \frac{m + 4\pi r^3 p}{r(r - 2m)} \quad (6)$$

The first equation follows from the vanishing of $T^{\mu\nu}_{;\nu}$, while the second and third equation are Einstein's equation written in the orthonormal reference frame with tetrads given by,

$$\mathbf{e}_t = \frac{\partial}{e^\Phi \partial t}, \mathbf{e}_{\hat{r}} = \frac{\partial}{e^\Lambda \partial r}, \mathbf{e}_{\hat{\theta}} = \frac{\partial}{r \partial \theta}, \mathbf{e}_{\hat{\phi}} = \frac{\partial}{r \sin \theta \partial \phi} \quad , \quad (7)$$

Equation (5) is equivalent to the constraint on initial data

$${}^3R = 16\pi\rho \quad (8)$$

where 3R is the scalar curvature of the intrinsic geometry defined by $\mathbf{e}_{\hat{r}}, \mathbf{e}_{\hat{\theta}},$ and $\mathbf{e}_{\hat{\phi}}$.

We will now show that two of these three equations can be derived from thermodynamics. Normally, the thermodynamical behavior of self-interacting theories can be problematic, but in the rest frame of the perfect fluid we can use all the laws of thermodynamics, and then exploit the principle of equivalence. In particular, we can use the Gibbs-Duhem relation, since this equation involves local quantities which are scalar fields. Equation (3) is therefore valid in all reference frames. It should be noted that while the system obeys the Gibbs-Duhem locally, it does not necessarily obey it globally [12].

Since the system is in thermal and chemical equilibrium, we can use the Tolman relation [14], which tells us that the temperature and chemical potential at any two points are related by the redshift factor. In terms of quantities at infinity T_o, μ_o , this gives us

$$T(r) = T_o e^{-\Phi(r)} \quad (9)$$

and

$$\mu(r) = \mu_o e^{-\Phi(r)} \quad . \quad (10)$$

These relations are purely geometrical and do not depend on Einstein's equations. The Tolman relation is simply a reflection of the fact that frequencies and energies get shifted by a factor equal to the norm of the static killing vector.

We can now derive equation (4) which (not surprisingly) follow directly from the first law of thermodynamics. While this result is trivial, the derivation we will employ is rather interesting in that we will use the thermodynamical relations (3), (9),(10), without using any other machinery of differential geometry. In other words, local extensivity, plus the Tolman relation encode all the necessary information. It is also instructive as it demonstrates the validity of locally defining the temperature and other thermodynamic quantities (which can be defined via the first law).

Combining equations (3), (9) and (10) we can write the entropy density as

$$s = \beta_o e^\Phi (\rho + p) - \mu_o \beta_o n \quad (11)$$

where β_o is just the inverse temperature at infinity. Taking the derivative of this equation with respect to r we find

$$\frac{ds}{dr} = \beta_o e^\Phi \left(\frac{d\rho}{dr} + \frac{dp}{dr} \right) + \beta_o \frac{d\Phi}{dr} e^\Phi (\rho + p) - \beta_o \mu_o \frac{dn}{dr} \quad (12)$$

and using the first law

$$d\rho = T ds - \mu dn \quad (13)$$

gives us

$$\frac{dp}{dr} = -(\rho + p) \frac{d\Phi}{dr} \quad (14)$$

as required.

We now derive one of Einstein's equations by extremizing the total entropy of the matter. We will perform the variation at fixed particle number N and fixed energy at infinity. For the latter, we will assume that the energy at infinity is given by the ADM mass

$$M = \int_0^\infty 4\pi r^2 \rho dr \quad (15)$$

This equation clearly has physical content, however, it is not equivalent to assuming the full Einstein's equation. It is a single integral equation and from it we will derive an equation for the metric which is valid throughout the space time i.e. from a single equation, we derive an infinite number of equations, one at every point in space. In our derivation, we don't need

to assume that the quantity given in equation (15) is actually the total energy, just that it is fixed during the variation. However, in order to make contact with thermodynamics, one would need to identify M with the total energy.

The quantity N is simply given by integrating the local particle number density n over the proper volume element $dV = 4\pi r^2 e^\Lambda dr$

$$N = \int_0^\infty n 4\pi r^2 e^\Lambda dr \quad , \quad (16)$$

Unlike equation (15), the equation for N does not rely on any assumptions other than the equivalence principle.

Before stating the theorems we will prove, we first introduce the following useful change of variables

$$\frac{dm}{dr} \equiv 4\pi r^2 \rho \quad . \quad (17)$$

This is simply a definition, and no assumptions need be made concerning the physical meaning of m . In particular, we do not assume that m is given by equation (5).

We now prove the following:

Theorem 1 *If we assume that Λ is an arbitrary function of m and r , then*

$$4\pi r^2(\rho + p) \frac{\partial \Lambda}{\partial m} = \frac{d\Lambda}{dr} + \frac{d\Phi}{dr} \quad (18)$$

This equation is a linear combination of of Einstein's equations (5) -(6) .

Corollary 1 *If Einstein's equations (5) and (6) are satisfied, then the entropy is an extremum.*

Corollary 2 *If the equation of constraint (8) holds and the entropy is an extremum, then the rest of Einstein's equations hold. Put another way, if we assume the tt component of Einstein's equations, which gives us Λ , then we can derive the rr component of Einstein's equations which gives us Φ .*

We now prove Theorem 1 by varying the total entropy of the system, to find the extremum.

Since s is the local entropy as measured by observers in the rest frame of the fluid, we can integrate this quantity over the proper volume to obtain the total entropy S .

$$S = \int_0^\infty s 4\pi r^2 e^\Lambda dr \quad (19)$$

We can then append equation (16) to the total entropy as a constraint on the total particle number by defining a new quantity L

$$L = S + \lambda \left(\int_0^\infty 4\pi r^2 n e^\Lambda dr - N \right) \quad (20)$$

The constraint on the total energy could also be appended as a constraint, however, this is not necessary as it arises quite naturally through the variation principle. Since Λ depends on m and r , we can simply vary it with respect to m , keeping the variation fixed at the end points. The vanishing of δm at the endpoints is equivalent to varying the entropy at fixed total energy.

We can proceed with the variation by treating the quantities $\rho(r)$ and $n(r)$ as the independent thermodynamical variables. We can then extremize the entropy, by varying it with respect to these quantities in much the same way as a Lagrangian is varied with respect to conjugate variables $q(t)$ and $\dot{q}(t)$. We find

$$\delta L = \int_0^R 4\pi r^2 dr e^\Lambda \left(\beta_o e^\Phi \delta \rho + (\lambda - \mu_o \beta_o) \delta n + (s(\rho, n) + \lambda n) \delta \Lambda \right) \quad (21)$$

where we have used the thermodynamical relations

$$\beta = \beta_o e^\Phi = \left(\frac{\partial s}{\partial \rho} \right)_n \quad (22)$$

and

$$\mu \beta = \mu_o \beta_o = - \left(\frac{\partial s}{\partial n} \right)_\rho \quad (23)$$

We can rewrite $\delta \rho$ from our definition of m given in equation (17)

$$\delta \rho = \delta \dot{m} / (4\pi r^2) \quad (24)$$

where $\dot{\cdot} \equiv \frac{d}{dr}$. Substituting this and equation (11) into (21) and then integrating by parts gives

$$\begin{aligned} \delta S = & \int_0^R dr \beta_o e^{\Lambda+\Phi} \left(4\pi r^2 (\rho + p) \delta \Lambda - (\dot{\Lambda} + \dot{\Phi}) \delta m \right) \\ & + \int_0^R 4\pi r^2 dr e^\Lambda (\lambda - \mu_o \beta_o) (\delta n + n \delta \Lambda) \end{aligned} \quad (25)$$

where we have held the variation fixed at the endpoints. As stated previously, the vanishing of $\delta m(R) = \delta M$ is equivalent to performing the variation at fixed energy, while the vanishing of $\delta m(0)$ is a necessary boundary condition to keep $\rho(0)$ finite. We can then substitute the relation

$$\delta\Lambda(m, r) = \frac{\partial\Lambda}{\partial m}\delta m \quad (26)$$

into Eq. (25) and since the system is in equilibrium, the variation of the total entropy must vanish. The variation is independent at each point r , and so, the vanishing of δL implies

$$\beta_o\mu_o = \lambda \quad (27)$$

and

$$4\pi r^2(\rho + p)\frac{\partial\Lambda}{\partial m} = \frac{d\Lambda}{dr} + \frac{d\Phi}{dr} \quad (28)$$

The first equation is consistent with the Tolman relation. Eq. (28) is the central result of this paper as articulated by Theorem 3. One can further verify that Eq. (28) is related to the linear combination of Einstein's equations given by

$$G_{\hat{t}\hat{t}} - 8\pi T_{\hat{t}\hat{t}} + G_{\hat{r}\hat{r}} - 8\pi T_{\hat{r}\hat{r}} = 0 \quad . \quad (29)$$

Corollary 1 is proved by verifying that Φ and Λ as obtained from Einstein's equations (5) and (6) satisfy Eq. (28). The entropy will therefore be an extremum. Corollary 2 can be verified by substituting the solution from the $G_{\hat{t}\hat{t}}$ component of Einstein's equation (Eq. 5) into Eq. (28) yielding the correct result for the $G_{\hat{r}\hat{r}}$ component of Einstein's equation (Eq. 6)

III. A THERMODYNAMIC SECTOR OF QUANTUM GRAVITY

We now see that for this system, we can recast Einstein's equations (5)-(6) into another set of equations, namely equation (28) which can be classified as thermodynamical in nature, and an additional equation representing the true physical theory. The choice of this latter equation is somewhat arbitrary. We could for example choose the constraint (Eq. (8)) to represent the true degrees of freedom. This equation completely determines Λ in terms of the matter fields. Φ is then determined by extremizing the entropy to find the most probable configuration. Quantization of the full system is then relatively straightforward, as the quantization of the gravitational variable Λ follows by reducing the phase space via Eq.

(8). The full Lagrangian is then given by integrating the Lagrangian density of the matter fields over the proper volume element

$$L = 4\pi \int \mathcal{L} r^2 e^{\Phi(r)} \sqrt{1 - 2\hat{m}/r} dr \quad (30)$$

where \hat{m} is the operator version of m defined via equation (17). Φ simply acts as a potential as far as the quantization of the matter fields go.

The feature that Λ is quantum in nature while Φ is purely thermodynamical results from the assumption that equation (8) gave the physical theory, thus breaking the symmetry between space and time.

We could also choose for the physical theory, a generally covariant equation such as the trace of Einstein's equations

$$R = \frac{16}{2-d} \pi T \quad . \quad (31)$$

Such an equation and its quantized version, have appeared at various times in the literature, particularly in the context of lower dimensional gravity [16]. The physical theory would then give rise to Eq. (28) through Theorem 1.

An analogy can be made between our recasting of Einstein's equations into two sectors, and the thermodynamics of an ideal gas in a box. Consider for example the derivation of equilibrium conditions for the gas. One can derive the condition that the pressure must be constant by imagining that the box is divided into two sections by a movable, heat conducting wall. The total volume is fixed, but the volume of each compartment can change as the wall moves. Extremizing the entropy then gives the conditions for mechanical equilibrium - namely, that the pressure in each section will be equal. The laws governing the individual gas molecules and walls of the box can be quantized in the usual way, but the equation for the equality of pressures should not be quantized. The latter equation is just a statement about the most probable configuration of the system.

Likewise, with our gravitating fluid, one might regard Eq. (28) as coming purely from thermodynamics. If this is the case, it should not be quantized.

Although one can divide Einstein's equations into a pure thermodynamic sector and a physical sector for this particular system, whether it is possible to do so more generally as one might conjecture is speculation. For a system of thin spherical shells, the conjecture holds [8], but more work is needed to ascertain how general these results are. Certainly systems which are far out of equilibrium (such as matter containing shock-waves) would be

difficult to analyze, since at present, not enough is known about non-equilibrium thermodynamics. However, even in the simple model we have considered, the fact that the entropy is extremized only if the combination of Einstein's equations given by (28) holds is rather puzzling, and does appear to demand an explanation.

IV. CONCLUSION

We have seen that the entropy of a spherically symmetric fluid in equilibrium is only an extremum when Einstein's equations are satisfied. Conversely, one can derive part of Einstein's equations from thermodynamical consideration. This allows us to recast Einstein's equations into a smaller physical sector which ought to be quantized. This physical sector then gives rise to equilibrium conditions which are equivalent to the rest of Einstein's equation. These latter equations ought not to be quantized as they are purely thermodynamical in nature. One can therefore speculate that in general not all of Einstein's equations should be quantized.

Speculation aside, the identity between Einstein's equations and thermodynamics for this system is very puzzling, and in many respects analogous to the connections that exist for black holes. Here however, we have no horizon, and so the entropy is unambiguously related to the number of states of the system. In the black-hole case, the entropy is usually associated with the gravitational degrees of freedom of the horizon. Furthermore, the connection appears very strong - the condition for thermodynamic equilibrium is equivalent to one of Einstein's equations. This might give some further insight into the thermodynamics of black holes where the connection between Einstein's equation and the entropy is more ambiguous.

In fact, one already sees from equations (19) and (11) that the total entropy S need not be purely volume scaling due to the presence of the metric functions. The total integrated entropy scales differently than the total energy as given by equation (15). This is similar to black holes, where the entropy is also not volume scaling (being proportional to the area). This effect was explored in reference [12], where it was shown that an equilibrium system which has an extensive entropy in the absence of gravity, will have an entropy proportional to its area in the limit that the system is about to form a black hole.

The fact that one can calculate the entropy for this system, and that this entropy is an extremum if Einstein's equations holds, is also of interest in terms of understanding the

thermodynamics of strongly interacting systems. It appears that one can learn much about gravitational systems simply by conducting the analysis in terms of local thermodynamical variables. It is hoped that the results here might lead to a greater understanding of other non-extensive systems.

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